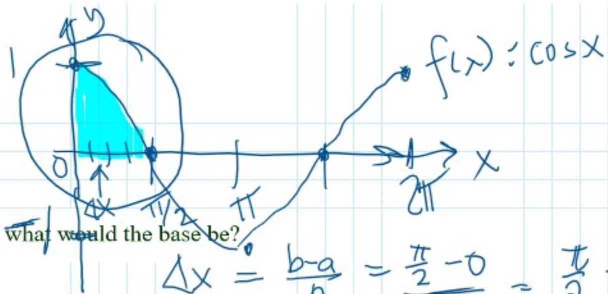
30-Jan/1-Feb 2023 Lecture 3

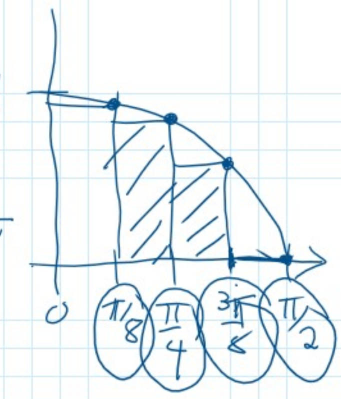
Sunday, January 29, 2023 11:08 AM

Do: Sketch $f(x) = \cos x$ on $\left[0, \frac{\pi}{2}\right]$ (a, b)



Do: if the region were split into 4 rectangles, what would the base be? (h, bh)

$\Delta x = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{4} = \frac{\pi}{2} \cdot \frac{1}{4} = \frac{\pi}{8}$



Do: what x-values would determine rectangles' heights for...

L4: 0, $\pi/8$, $\frac{\pi}{4}$, $\frac{3\pi}{8}$, $\frac{\pi}{2}$ (Δx)

R4: 0, $\pi/8$, ...

M4: 0, $\pi/16$, $\frac{3\pi}{16}$, $\frac{5\pi}{16}$, $\frac{7\pi}{16}$

x_1, x_2, x_3, x_4

$\frac{\pi}{16} + \frac{2}{2}\frac{\pi}{8} = \frac{3\pi}{16}$

review x_i^* $\quad x_1^*$ x_2^* ...

going forward, $x_i^* \rightarrow x_i$

ex. determine function/interval whose area is equal to ($f(x)$, $[a,b]$)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\frac{2}{n}}_{b=\Delta x} \underbrace{\left(5 + \frac{2i}{n}\right)^{10}}_{f(x_i)=h}$$

$a + \Delta x \cdot i = x_i$

$\Delta x = \frac{b-a}{n} = \frac{2}{n}$

$\frac{b-5}{n} = \frac{2}{n} \Rightarrow b - 5 = 2$, $b = 7$ $\qquad [5,7]$

$(x_i)^{10} \Rightarrow f(x) = x^{10}$ on $[5,7]$

WORKING BACKWARDS ... a, b

ex. given $f(x) = \cos x_i$ on $\left[0, \frac{\pi}{2}\right]$

use definition of R_n

to express area using limits and summation notation (sigma)

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \cdot \cos(a + \Delta x \cdot i)$$

$$\frac{\pi}{2n} \cdot \cos\left(0 + \frac{\pi}{2n} i\right)$$

$\Delta x = ?$
$a = ?$

$\Delta x = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{n} = \frac{\pi}{2n}$

$$\boxed{A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{2n} \cdot \cos\left(\frac{\pi}{2n} i\right)}$$

ex. Runner's speed increased steadily during first three seconds of race. Find upper and lower estimates for distance traveled in three seconds. Round to 1 decimal place.

increment (Δx) = .5

t	0	.5	1	1.5	2	2.5	3
v	0	6.2	10.8	14.9	18.1	19.4	20.2

1 2 3 4 5 6

Upper: $R_6 = .5(6.2 + 10.8 + 14.9 + 18.1 + 19.4 + 20.2) = 44.8$

Lower: $L_6 = .5(0 + 6.2 + 10.8 + 14.9 + 18.1 + 19.4) = 34.7$

Riemann Sum

Section 5.2: Definite Integral

Riemann Sum

If f is defined on $[a,b]$ then $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot\Delta x$ where:

- $\Delta x = \frac{b-a}{n}$
- $x_i = a + \Delta x \cdot i$

$$\int_a^b f(x)\,dx$$

limits or bounds: a = lower bound, b = upper bound

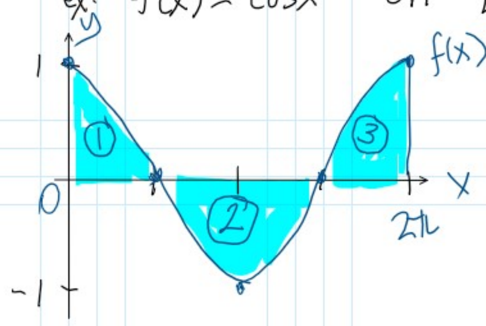
$f(x)$: integrand

dx: defines variable for integration

$\int$: elongated S (summation) integral sign

functions can dip below x-axis:

ex. $f(x) = \cos x$ on $[0, 2\pi]$

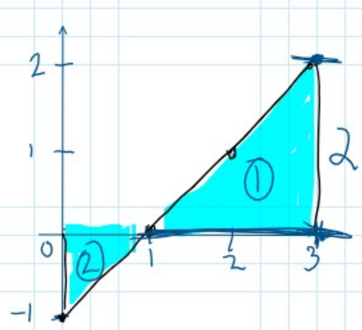


When curve dips below x-axis i.e. negative y-values, subtract area

$A_{\text{NET AREA}}$ (~~TOTAL~~) $= A_1 - A_2 + A_3$

Do: calculate net area under $f(x) = x-1$ on $[0,3]$

answer: $\frac{3}{2}$



net area $= A_1 - A_2$ $\qquad A_\Delta = \frac{1}{2}bh$

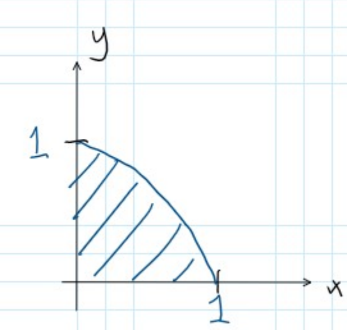
$= \frac{1}{\cancel{2}}\cancel{2}(2) - \frac{1}{2}(1)(1)$

$= 2 - \frac{1}{2} = \frac{3}{2}$

CAN ALSO BE WRITTEN AS: $\int_0^3 (x-1)\,dx$

$\therefore$ definite integral can be interpreted as a net area

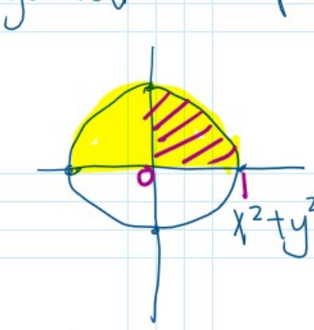
ex. evaluate $\int_0^1 \sqrt{1-x^2}\,dx$ by interpreting as a geometric shape $\xrightarrow{\text{pic}}$



$y^2 = \sqrt{1-x^2}^{\,2}$

$y^2 = 1-x^2$

$x^2+y^2 = 1$ (r^2) circle w $r=1$ center $(0,0)$



$A_O = \pi r^2 \quad r=1$

$= \pi \cdot 1^2$

$= \pi$

$A_D = \frac{1}{4}\pi$ or $\boxed{\frac{\pi}{4}}$

~~ex. express as integral~~ $\quad \int_{-1}^{1} \sqrt{1-x^2}\,dx$

$\frac{1}{2}\pi$

ex. express $\lim_{n\to\infty} \sum_{i=1}^{n} \left((x_i)^3 + x_i \cdot \sin(x_i)\right)\Delta x$ as an integral on $[0, 3\pi]$ (a = 0, b = 3π)

$$\int_0^{3\pi} \left(x^3 + x\sin x\right) dx$$

$$\int_0^{3\pi}\left(x^3 + x\sin x\right)dx \qquad a \quad b$$

REVIEW SIGMA NOTATION

ex. evaluate $\int_0^3 \underbrace{(x^3-6x)}_{f(x)}\,dx$ using limits and sigma notation (summation)

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \boxed{\frac{3}{n}}$$

$$x_i = a + \Delta x \cdot i = 0 + \frac{3}{n}i = \frac{3i}{n} = x_i$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\cdot\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left((x_i)^3 - 6\cdot x_i\right)\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right)\cdot\frac{3}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{27i^3}{n^3} - \frac{18i}{n}\right)\cdot\frac{3}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{81i^3}{n^4} - \frac{54i}{n^2}\right)$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{81i^3}{n^4} - \sum_{i=1}^{n}\frac{54i}{n^2}\right]$$

$$= \lim_{n\to\infty}\left[\frac{81}{n^4}\sum_{i=1}^{n} i^3 - \frac{54}{n^2}\sum_{i=1}^{n} i\right]$$

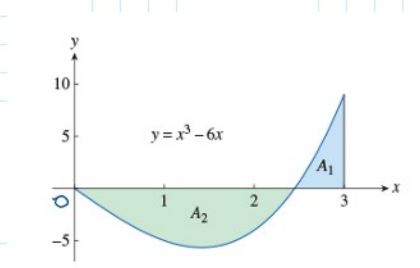
$$= \lim_{n\to\infty}\left[\frac{81}{n^4}\left(\frac{n(n+1)}{2}\right)^2 - \frac{\cancel{54}}{n^{\cancel{2}}}\cdot\frac{n(n+1)}{\cancel{2}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{81}{\cancel{n^4}_{n^2}}\cdot\frac{\cancel{n^2}(n+1)^2}{4} - 27\cdot\frac{n+1}{n}\right]$$

$$= \frac{81}{4}\lim_{n\to\infty}\frac{n^2+2n+1}{n^2} - 27\lim_{n\to\infty}\frac{n+1}{n}$$

$$= \frac{81}{4} - 27\cdot\frac{4}{4} = \frac{81}{4} - \frac{108}{4} = \boxed{-\frac{27}{4}}$$

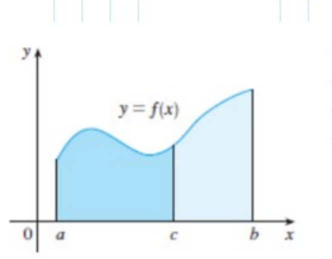
$$\lim_{x\to\infty} 3x^2 = 3\lim_{x\to\infty} x^2$$



area mostly neg ✓

PROPERTIES OF DEFINITE INTEGRAL

refer to PowerPoint templates



ex.
~~Do~~: given

f is an antiderivative of f'

Review: Antiderivatives If f' is a derivative then f is an antiderivative of f'

Notation: F is an antiderivative of f on some interval I
if $F'(x) = f(x)$ for all x on I

Most General Antiderivative: If F is an antiderivative of f on I
then the most general antiderivative of f on I
$F(x) + C$

Common Antiderivatives NOT A COMPLETE LIST!

deriv $\rightarrow$ $f(x) = \cos x$
$F(x) = \sin x + C$

$f(x) = e^x$ deriv
$F(x) = e^x + C$

$f(x) = \frac{1}{x}$
$F(x) = \ln|x| + C$

Power Function - Antiderivative $(ax^n)' = anx^{n-1}$

derivative

$$\boxed{\begin{aligned} f(x) &= x^n \\ F(x) &= \frac{x^{n+1}}{n+1} + C \end{aligned}}$$

antiderivative

ex. find the most general antiderivative of $f(x) = 8x^9 - 3x^6 + \frac{10}{x^9}$
[answer should contain only positive exponents]

$\frac{10}{x^9} = 10x^{-9}$

$$F(x) = 8\frac{x^{10}}{10} - \frac{3x^7}{7} \cancel{+} - \frac{10x^{-8}}{8} + C$$
$$= \frac{4}{5}x^{10} - \frac{3}{7}x^7 - \frac{5}{4x^8} + C$$

SECTION 5.3 EVALUATING DEFINITE INTEGRALS

Revisit Indefinite Integral no bounds, family of solutions

$\Updownarrow$

General Antiderivative

POWER FUNCTION

$$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + C \quad \text{where } n \neq -1$$

$\int ax^n\,dx =$

ex. $\int \sqrt{x}\,dx = \int x^{1/2}\,dx$
$= \frac{x^{3/2}}{3/2} + C = \boxed{\frac{2}{3}x^{3/2} + C}$

$= \frac{x^{3/2}}{\frac{3}{2}} + C = \boxed{\frac{2}{3}x^{3/2} + C}$

ex. $\int \frac{1}{x}\,dx = \boxed{\ln|x| + C}$

(next slide)

ex. $\int (10x^4 - 2\sec^2 x)\,dx$

$= \frac{10}{5}x^5 - 2\int \sec^2 x\,dx$

$= \left(2x^5 - 2\tan x + C\right)'$ check w differentiation

$10x^4 - 2\sec^2 x + 0$

Section 5.3 - Evaluating Definite Integrals

Definite Integral has bounds from $x=a$ to $x=b$; there is a single solution

Evaluation Theorem if f is continuous on $a \le x \le b$ and $F'(x) = f(x)$ then $\int_a^b f(x)\,dx = F(b) - F(a)$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Do: $\int_0^3 (x^3 - 6x)\,dx = \boxed{-\frac{27}{4}}$

$= \left[\frac{x^4}{4} - \frac{6x^2}{2}\right]_0^3$

$= \underbrace{\left(\frac{x^4}{4} - 3x^2\right)}_{F(x)}\Big|_0^3$

$= \frac{1}{4}(3^4 - \cancel{0^4}) - 3(3^2 - \cancel{0^2})$

$= \frac{1}{4}81 - 3(9)$

$= \frac{81}{4} - 27 \cdot \frac{4}{4} = \boxed{-\frac{27}{4}}$

Do: $\int_0^{\pi/2} \cos x\,dx = \boxed{1}$

$\int_0^{\pi/2} \cos x$

$= \sin x \Big|_0^{\pi/2}$

$= \sin\frac{\pi}{2} - \sin 0$

$= 1 - 0$

$= \boxed{1}$

$\sqrt{x^2} = x$

Net Change Theorem := integral of a rate of change is the net change

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

f — $s(t)$ position
f' — $v(t)$ velocity
f'' — $a(t)$ acceleration

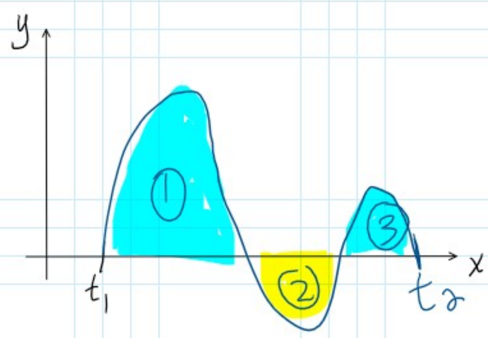
ex. if $V(t)$ is volume of a lake at time t then its derivative $V'(t)$ is the rate at which water flows into lake at t: i.e., $\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$

ex. object moves along a straight line w position, $s(t)$

$v(t) = s'(t) \Rightarrow \int_{t_1}^{t_2} v(t)\,dt = \underbrace{s(t_2) - s(t_1)}$,

result is net change in position or displacement

to find total distance traveled, take absolute value



DISPLACEMENT = $\int_{t_1}^{t_2} v(t)\,dt = A_1 - A_2 + A_3$

vs.

DISTANCE = $\int_{t_1}^{t_2} |v(t)|\,dt = A_1 + A_2 + A_3$

ex. A particle moves along a straight line such that its velocity can be represented as $v(t) = t^2 - t - 6$ m/s on $1 \le t \le 4$

a. find displacement $\int_1^4 v(t)\,dt = \int_1^4 (t^2 - t - 6)\,dt$

$= \left(\frac{t^3}{3} - \frac{t^2}{2} - 6t\right)\Big|_1^4 = V(4) - V(1)$

$= \frac{1}{3}(4^3 - 1^3) - \frac{1}{2}(4^2 - 1^2) - 6(4-1)$

$= \boxed{-\frac{9}{2}\text{ m}}$

b. find total distance $v(t) = t^2 - t - 6 = 0$

$(t-3)(t+2) = 0$

cp: $t = 3$ $t = -2$

	-2	1		3		4
$t-3$			$-$	0	$+$	
$t+2$	0		$+$		$+$	
$(t-3)(t+2)$			$-$		$+$	

$= -\int_1^3 (t^2 - t - 6)\,dt + \int_3^4 (t^2 - t - 6)\,dt$

$= -\left(\frac{t^3}{3} - \frac{t^2}{2} - 6t\right)\Big|_1^3 + \left(\frac{t^3}{3} - \frac{t^2}{2} - 6t\right)\Big|_3^4$ v'

$= \boxed{\frac{61}{6}\text{ m}}$